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Wave number of the coherent acoustic field in a medium with randomly distributed spheres

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Abstract
This paper presents an iterative method for numerically solving the secular equation obtained by Fikioris and Waterman for the effective wave number of the coherent acoustic field propagating in a medium with a random distribution of identical spherical scatterers. The method works both for the original equation derived by Fikioris and Waterman and for its generalization to the case of an arbitrary two-point correlation function in the positions of any two scatterers. An explicit solution up to second order in the density of scatterers is also obtained. In the point scatterer limit this solution is identical to that obtained by Lloyd and Berry which is considered to be the correct result for the effective wave number to second order in the density of scatterers in the point scatterer limit.

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1. Introduction

Fikioris and Waterman in [1] followed up the work of Waterman and Truell in [2] by incorporating the requirement of non-superposition of scatterers into the problem of the transmission and reflection of an acoustic plane wave normally incident upon a half-space containing randomly distributed identical spheres. These authors obtained an integral equation for the coherent field by a statistical assumption, which is similar to the quasi-crystalline approximation (QCA, see [3]). From this equation, these authors derive a secular equation for the effective wave number in the composite medium formed by the randomly distributed scatterers. As the QCA is known to include all double-scattering processes as shown by Henyey in [4] for the case of the Foldy approximation (which is analogous to the QCA, see [5]) and since the Lloyd and Berry [6] wave number was obtained from a multiple scattering expansion that explicitly included all the double scattering processes one would expect, on

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consistency grounds, that the Lloyd–Berry wave number [6] should be obtainable from the Fikioris–Waterman secular equation ([1]). This is indeed the case as will be shown in this paper. The Lloyd–Berry effective wave number contains the first three terms in an expansion in powers of the density of scatterers in the point scatterer limit and it is considered to be the correct result under those conditions. These previous results ignore any further two-point correlations between the scatterer's positions except for the non-superposition condition. In this paper, it is shown how to incorporate an arbitrary correlation function into the Fikioris–Waterman secular equation. In section 2, the result of Fikioris and Waterman is presented without derivation and the notation is established. In section 3, the secular equation for the effective wave number is extended to accommodate arbitrary pair correlations between the scatterer's positions. In section 4, an implicit equation for the effective wave number is derived and it is shown how an iterative method can be used to obtain this wave number. In section 5, an expansion of the effective wave number is obtained up to second-order terms in the density of scatterers. In the point scatterer limit this expression yields the Lloyd–Berry wave number. Finally, section 6 presents a summary of this paper including additional comments and conclusions.

2. The Fikioris–Waterman secular equation

The notation in the article by Fikioris and Waterman will be followed in this section. These authors derive the following equation, equation (2.12) in their article [1], for the amplitude of a coherent wave excited by the normally incident plane wave in the half-space ($z \geqslant 0$) occupied by the identical scattering spheres with density n_0:

$$A_n^0 = n_0 \sum_{j=0}^{\infty} (2j+1) B_j A_j^0 \sum_{p=0}^{\infty} (-\mathrm{i})^p a(0, j|0, n|p) d_p(k, K|b). \tag{1}$$

In the above equation, j, n and p are partial-wave indices, the scattering amplitude of each sphere is written as

$$f(\theta) = \frac{1}{\mathrm{i}k} \sum_{j=0}^{\infty} (2j+1) B_j P_j(\cos(\theta)), \tag{2}$$

and the coefficients $a(0, j|0, n|p)$ are defined by

$$\begin{aligned} P_j(x) P_n(x) &= \sum_{p=0}^{\infty} a(0, j|0, n|p) P_p(x), \\ a(0, j|0, n|p) &= \frac{2p+1}{2} \int_{-1}^{1} \mathrm{d}x\, P_j(x) P_n(x) P_p(x). \end{aligned} \tag{3}$$

The quantity $d_p(k, K|b)$, a consequence of imposing the non-superposition correlation $H(|\mathbf{r}_i - \mathbf{r}_j| - b)$ ($H(x)$ is the Heaviside step function, $b = 2a$ is the diameter of the spheres) between scatterers at $\mathbf{r}_i$ and $\mathbf{r}_j$, can be written as

$$d_p(k, K|b) = -\frac{4\pi b^2}{K^2 - k^2} \mathrm{i}^p [k h_p'(kb) j_p(Kb) - K h_p(kb) j_p'(Kb)], \tag{4}$$

where $h_p(kb)$ is the spherical Hankel function of the first kind and K is the effective wave number in the half-space occupied by the scatterers.

The above equation for A^0 (equation (1)) is a linear, homogeneous, algebraic equation of the form $[I - M(K)]\ A^0 = 0$. A solution $A^0 \neq 0$ can only exist for the particular value of

K that makes the matrix $I - M(K)$ singular. The (infinite-dimensional) matrix $M(K)$ will be written as

$$M(K)_{n,j} = \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)}(2j+1)B_j \sum_{p=0}^{\infty} a(0, j|0, n|p) J_p(k, K|b), \tag{5}$$

where

$$J_p(k, K|b) = -\mathrm{i}kb^2[kh'_p(kb)j_p(Kb) - Kh_p(kb)j'_p(Kb)], \qquad J_p(k, k|b) = 1. \tag{6}$$

Thus the effective wave number is obtained as the value of K such that the Fikioris–Waterman secular equation is satisfied:

$$\det(I - M(K)) = 0. \tag{7}$$

3. Extension to arbitrary pair correlations

In this section, the Fikioris–Waterman equation (equation (1)) will be extended to the case of an arbitrary pair correlation function between the positions of any two scatterers in the ensemble. As is well known a random distribution of identical spheres will exhibit correlations among the positions of the spheres, which become stronger as the density increases. The estimation of those correlations is entirely analogous to the computation of initial-state correlations in the classical–mechanical models of fluids (see [7]). For the pair correlation, $g_2(|\mathbf{r} - \mathbf{r}'|)$, one has the following properties:

$$g_2(r) \xrightarrow[r\to\infty]{} 0 \qquad \text{and} \qquad g_2(r) = 1 \qquad \text{for} \quad r < b, \quad b = \text{sphere diameter}. \tag{8}$$

The condition $g_2(r \to \infty) = 0$ obviously indicates the expectation that the positions of widely separated scatterers are not correlated while the condition $g_2(r) = 1$ enforces the physical requirement that two scatterers should not overlap. A simple approximation to $g_2(|\mathbf{r} - \mathbf{r}'|)$ is

$$1 - g_2(|\mathbf{r} - \mathbf{r}'|) = H(|\mathbf{r} - \mathbf{r}'| - b), \tag{9}$$

which is equivalent to the Fikioris–Waterman hole correction. The hole correction is responsible for the appearance of the quantity $J_p(k, K|b)$ on the right-hand side of equation (4). From the derivation of equation (1) given in [1] one finds that $J_p(k, K|b)$ can be written as

$$J_p(k, K|b) = -\mathrm{i}k \int_0^\infty \mathrm{d}r\, H(r - b)\partial_r\{r^2[h_p(kr)\partial_r j_p(Kr) - j_p(Kr)\partial_r h_p(kr)]\}. \tag{10}$$

Thus the generalization for an arbitrary pair correlation is

$$J_p(k, K|b) = -\mathrm{i}k \int_0^\infty \mathrm{d}r[1 - g_2(r)]\partial_r\{r^2[h_p(kr)\partial_r j_p(Kr) - j_p(Kr)\partial_r h_p(kr)]\}. \tag{11}$$

Using the property $g_2(r) = 1$ for $r < b$ one can write

$$\begin{aligned} J_p(k, K|b) = {} & \mathrm{i}kb^2[1 - g_2(b^+)][h_p(kr)\partial_r j_p(Kr) - j_p(Kr)\partial_r h_p(kr)]_{r=b} \\ & - \mathrm{i}k \int_{b^+}^\infty r^2\, \mathrm{d}r[\partial_r g_2(r)][h_p(kr)\partial_r j_p(Kr) - j_p(Kr)\partial_r h_p(kr)]. \end{aligned} \tag{12}$$

By b^+ it is meant that the value of $g_2(r)$ at $r = b$ is reached from the region $r > b$.

Therefore, for the remainder of this paper, the matrix $M(K)$ is still defined by equation (5) but $J_p(k, K|b)$ is given by equation (11) or its equivalent equation (12) and one still has $J_p(k, k|b) = 1$ since the expression involving spherical Bessel and Hankel functions in equation (12) reduces to the Wronskian when $K = k$.

4. The implicit equation for the effective wave number

Instead of attempting to solve the infinite-dimensional secular equation $\det(I - M(K)) = 0$ the condition $A^0 \neq 0$ will be used to obtain an implicit equation for the effective wave number K. Now, with $J_p(k, k|b) = 1$, one can write $M(K) = M^0(K) + R(K)$ where

$$M^0(K)_{n,j} = \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)}(2j+1)B_j \sum_{p=0}^{\infty} a(0, j|0, n|p) = \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)}(2j+1)B_j, \tag{13}$$

since $\sum_{p=0}^{\infty} a(0, j|0, n|p) = P_j(1)P_n(1) = 1$. For $R(K)$ one has

$$\begin{aligned} R(K)_{n,j} &= \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)}(2j+1)B_j S(K)_{n,j}, \\ S(K)_{n,j} &= \sum_{p=0}^{\infty} a(0, j|0, n|p)[J_p(k, K|b) - 1]. \end{aligned} \tag{14}$$

One notices that $K^2 - k^2$ and $J_p(k, K|b) - 1$ are of order n_0. Thus, to lowest order, $M^0(K)$ is independent of n_0 and $R(K)$ is of order n_0. From the equation $[I - M(K)]A^0 = 0$ one can extract an implicit equation for the effective wave number. First one observes that

$$A_n^0 = [M(K)A^0]_n \Rightarrow A_n^0 = \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)} U \sum_j \left(\frac{I}{I - R(K)}\right)_{n,j}, \tag{15}$$

where

$$U = \sum_j (2j+1)B_j A_j^0. \tag{16}$$

Entering the expression for A_j^0 from equation (15) into the definition of U in equation (16) one obtains

$$U = \frac{4\pi n_0}{\mathrm{i}k(K^2 - k^2)} U \sum_l \sum_j (2j+1)B_j \left(\frac{I}{I - R(K)}\right)_{j,l}, \tag{17}$$

demanding that $U \neq 0$ leads to

$$K^2 = k^2 + 4\pi n_0 F(K), \tag{18}$$

with

$$F(K) = \frac{1}{\mathrm{i}k} \sum_j (2j+1)B_j \sum_l \left(\frac{I}{I - R(K)}\right)_{j,l}. \tag{19}$$

Equation (18) is an implicit equation for the effective wave number that can be solved iteratively. The effective wave number can be obtained by setting up the following iteration scheme:

$$K_{m+1}^2 = k^2 + 4\pi n_0 F(K_m), \qquad K_0^2 = k^2 + 4\pi n_0 f(0). \tag{20}$$

Note that from equation (19) it follows that one needs only the sum $Y_j(K) = \sum_l [I/(I - R(K))]_{j,l}$. Thus, in order to obtain $F(K)$ with a given K, instead of having to compute an inverse matrix one just has to solve a linear algebraic equation for $Y_j(K)$:

$$\sum_{n=0}^{\infty} [I - R(K)]_{j,n} Y_n(K) = 1, \qquad F(K) = \frac{1}{\mathrm{i}k} \sum_{j=0}^{\infty} (2j+1)B_j Y_j(K). \tag{21}$$

5. Density expansion for the effective wave number

In this section, a density expansion for the square of the effective wave number will be obtained explicitly to order n_0^2. In the point scatterer limit, this expression is identical to that obtained by Lloyd and Berry [6]. The expression for the effective wave number to order n_0^2 is obtained by noticing that the coupling matrix in the equation for $Y_j(K)$ (equation (21)) is of order n_0 and so $Y_j(K) = 1 + \mathcal{O}(n_0)$. To determine $Y_j(K)$ to first order in n_0 one observes that

$$\lim_{K\to k} \frac{4\pi n_0}{\mathrm{i}k(K^2-k^2)} S_{j,n}(K) = \frac{2\pi n_0}{\mathrm{i}k^2}[\partial_K S_{j,n}(K)]_{K=k}, \tag{22}$$

since $K = k + \mathcal{O}(n_0)$ and $S_{j,n}(k) = 0$. Therefore

$$K^2 = k^2 + 4\pi n_0 f(0) - \frac{2}{k}\left(\frac{2\pi n_0}{k}\right)^2 \sum_j (2j+1)B_j \sum_{n=0}^{\infty}(2n+1)B_n[\partial_K S_{j,n}(K)]_{K=k} + \mathcal{O}\big(n_0^3\big). \tag{23}$$

Now,

$$[\partial_K S_{j,n}(K)]_{K=k} = \int_{-1}^{1} \mathrm{d}x\, P_j(x)P_n(x) \sum_{p=0}^{\infty} \frac{2p+1}{2} P_p(x)[\partial_K J_p(k, K|b)]_{K=k}, \tag{24}$$

and one obtains

$$\sum_j (2j+1)B_j \sum_{n=0}^{\infty}(2n+1)B_n[\partial_K S_{j,n}(K)]_{K=k} = -\frac{k}{2}\int_{-1}^{1}\mathrm{d}x\, f(\arccos(x))^2 V(x), \tag{25}$$

with

$$V(x) \equiv k\sum_{p=0}^{\infty}(2p+1)P_p(x)[\partial_K J_p(k, K|b)]_{K=k}. \tag{26}$$

Thus, combining equations (23)–(26) yields

$$K^2 = k^2 + 4\pi n_0 f(0) + \left(\frac{2\pi n_0}{k}\right)^2 \int_{-1}^{1}\mathrm{d}x\, f(\arccos(x))^2 V(x) + \mathcal{O}\big(n_0^3\big). \tag{27}$$

The result in equation (27) is the second order in density expression for the effective wave number with finite-size spheres with pair-correlation $g_2(r)$. With $J_p(k, K|b)$ given by equation (12) one obtains

$$\partial_K J_p(k, K|b)_{K=k} = -\mathrm{i}b[1 - g_2(b^+)]B_p(kb) + \mathrm{i}\int_{b^+}^{\infty} r\,\mathrm{d}r[\partial_r g_2(r)]B_p(kr), \tag{28}$$

where

$$B_p(x) \equiv x^2 j_p'(x)h_p'(x) + xh_p(x)j_p'(x) + (x^2 - p(p+1))h_p(x)j_p(x). \tag{29}$$

The Lloyd–Berry effective wave number is obtained from equations (26) and (27) in the limit of uncorrelated point scatterers, that is, by taking $g_2(|\mathbf{r}-\mathbf{r}'|) = 1 - H(|\mathbf{r}-\mathbf{r}'| - b)$ and then the limit $b \to 0^+$ of equation (12) and using the result in equation (27). In this limit, $g_2(|\mathbf{r}-\mathbf{r}'|) = 0$ for $|\mathbf{r}-\mathbf{r}'| > 0$ and $J_p(k, K|b)$ is given by equation (6). One finds that

$$J_p(k, K|0) = \left(\frac{K}{k}\right)^p \quad \text{and} \quad \partial_K J_p(k, K|0) = \frac{p}{k}\left(\frac{K}{k}\right)^{p-1}. \tag{30}$$

Therefore, in the point scatterer limit,

$$V(x) = \sum_{p=0}^{\infty}(2p+1)pP_p(x) = \big(2\partial_z^2 + 3\partial_z\big)\left.\frac{1}{\sqrt{1-2xz+z^2}}\right|_{z=1}, \tag{31}$$

where the generating function for Legendre polynomials was used to obtain the last expression in equation (31). The last integral in equation (27), with $\mathcal{F}(x) = f(\arccos(x))^2$, is of the form

$$\int_{-1}^{1} \mathrm{d}x\, V(x)\mathcal{F}(x) = \left\{\left(2\partial_z^2 + 3\partial_z\right)\int_{-1}^{1} \mathrm{d}x \frac{1}{\sqrt{1-2xz+z^2}}\mathcal{F}(x)\right\}_{z=1}. \tag{32}$$

Integrating by parts on the right-hand side of equation (32) one obtains

$$\int_{-1}^{1} \mathrm{d}x\, V(x)\mathcal{F}(x) = \mathcal{F}(-1) - \mathcal{F}(1) + \int_{-1}^{1} \mathrm{d}x \sqrt{\frac{2}{1-x}}\partial_x \mathcal{F}(x). \tag{33}$$

In terms of $\theta = \arccos(x)$ one has

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}x\, V(x)\mathcal{F}(x) &= \int_{0}^{\pi} \sin(\theta)\,\mathrm{d}\theta[V(\cos(\theta))\mathcal{F}(\cos(\theta))] \\ &= \mathcal{F}(\cos(\theta))|_0^{\pi} - \int_0^{\pi} \mathrm{d}\theta \frac{\partial_\theta \mathcal{F}(\cos(\theta))}{\sqrt{\sin(\theta/2)}}.\end{aligned} \tag{34}$$

Therefore, one obtains the Lloyd–Berry [6] expression for the effective wave number to second order in the density of scatterers in the point scatterer limit:

$$K^2 = k^2 + 4\pi n_0 f(0) + \left(\frac{2\pi n_0}{k}\right)^2 \left\{f(\pi)^2 - f(0)^2 - \int_0^{\pi} \mathrm{d}\theta \frac{\partial_\theta[f(\theta)^2]}{\sqrt{\sin(\theta/2)}}\right\} + \mathcal{O}\left(n_0^3\right). \tag{35}$$

6. Summary

The main point of this paper was to derive the implicit equation for the effective wave number including pair correlations among the scatterer's positions and obtain its solution to second order in the density of scatterers. This is an explicit expression that is amenable to a relatively simple numerical calculation and thus applicable to the analysis of experiments involving transmission through a region occupied by a random distribution of scatterers, for example, transmission through a plane slab containing such scatterers. From the point of view of applications it is worthwhile to observe that it avoids having to compute numerically a very large (strictly speaking infinite-dimensional) determinant and then finding roots of the associated secular equation. For an example where this is done (truncating the dimensions of the determinant) see the work of Tsang *et al* in [8]. This reference also has a good discussion of the need to incorporate pair correlations when using the QCA at very high densities. The implicit wave-number equation (equation (18)) and its second order in density solution (equation (27)) are new results. Reproducing the result of Lloyd and Berry [6] from the general second-order solution indicates the correctness of the result, at least to second order in the density of scatterers. There are many published works on multiple scattering. A thorough review of those works would far exceed the scope of this paper. An encyclopedic review of wave propagation in random media and multiple scattering can be found in a book by Ishimaru [9]. A very useful review of multiple scattering methods in acoustics is found in the review article by Tourin *et al* [10].

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